A Mean Field Game of Controls: Closing The Loop of Optimal Trading

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Journée des Chaires ILB, October 2017
Optimal Trading And Mean Field of Controls

Optimal Trading [Lehalle et al., 2013, Chapter 3]

- Investors use trading algorithms to buy and sell large amounts of shares or contracts
- It meets the demand of regulators: more tractability, less complex products
- Intermediaries themselves use trading algorithms.

No more isolating an agent

- Up to now, the literature focused on one large investor facing a background noise (with the exception of [Jaimungal and Nourian, 2015] and [Firoozi and Caines, 2016], modeling one large risk-averse agent vs. small agents sensitive to their expected gain only).
- Here instead of having one isolated mean-variance agent [Almgren and Chriss, 2000],
- We will model all agents conducting simultaneously the same kind of strategies à la [Cartea and Jaimungal, 2015].
A Little Bit More on Mean Field Games in Finance

Generic considerations on MFG

▶ Mean Field Game is about a **continuum of agents**, characterized by their **distribution**,
▶ Each agent is fully identified by its position in the state space (from the viewpoint of one specific agent, others can be reordered),
▶ Each agent is sensitive to others via a **Mean Field**, and each agent contributes to this mean field (think about the pressure in a room where agents are particules),
▶ Each agent solves a (backward) optimization problem (his cost function can be a functional of the distribution at \( t \)),
▶ The distribution of agents is transported (via the controls) a forward way.

The natural mean field of financial markets

▶ Endogenous liquidity is often missing in the cost function of each agent (think about replicating bank’s risk),
▶ Each bank is facing a **mean field**, i.e. the aggregation of others’ actions is meant to be martingale,
▶ In reality **banks do communicate** via the global state of liquidity.
▶ **Liquidity is the natural mean field** to inject mathematical finance models in a game theoretical framework (slow: [Carmona et al., 2013] and HF: [Lachapelle et al., 2016], now instantaneous).
1. Standard Algorithmic Trading
2. Closing The Loop
3. An Explicit Solution For Identical Preferences
4. Learning The Mean Field
5. Qualitative Understanding (For Identical Preferences)
1 Standard Algorithmic Trading

- Closing The Loop
- An Explicit Solution For Identical Preferences
- Learning The Mean Field
- Qualitative Understanding (For Identical Preferences)
On our database of 300,000 large orders [Bacry et al., 2015]

Optimal Trading is about

▶ Trading slow enough to avoid market impact
▶ and fast enough so that the price is close to the decision.

Investors

▶ tackle decisions based on private information and portfolio construction methods,
▶ concentrate their decisions on their dealing desk,
▶ who study the liquidity of the portfolios to buy and sell,
▶ and use brokers to execute an automated way these decisions.
## Trading Algorithms: Typical Features

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Type of stock</th>
<th>Type of trade</th>
<th>Main feature</th>
</tr>
</thead>
<tbody>
<tr>
<td>PoV</td>
<td>Medium to large market depth</td>
<td>(1) Long duration position</td>
<td>(1) Follows current market flow, (2) Very reactive, can be very aggressive, (3) More price opportunity driven if the range between the max percent and min percent is large</td>
</tr>
<tr>
<td>VWAP / TWAP</td>
<td>Any market depth</td>
<td>(1) Hedging order, (2) Long duration position, (3) Unwind tracking error (delta hedging of a fast evolving inventory)</td>
<td>(1) Follows the “usual” market flow, (2) Good if market moves with unexpected volumes in the same direction as the order (up for a buy order), (3) Can be passive</td>
</tr>
<tr>
<td>Implementation Shortfall (IS)</td>
<td>Medium liquidity depth</td>
<td>(1) Alpha extraction, (2) Hedge of a non-linear position (typically Gamma hedging), (3) Inventory-driven trade</td>
<td>(1) Will finish very fast if the price is good and enough liquidity is available, (2) Will “cut losses” if the price goes too far away</td>
</tr>
<tr>
<td>Liquidity Seeker</td>
<td>Poor a fragmented market depth</td>
<td>(1) Alpha extraction, (2) Opportunistic position mounting, (3) Already split / scheduled order</td>
<td>(1) Relative price oriented (from one liquidity pool to another, or from one security to another), (2) Capture liquidity everywhere, (3) Stealth (minimum information leakage using fragmentation)</td>
</tr>
</tbody>
</table>
## Trading Algorithms: Typical Uses

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>Region of preference</th>
<th>Order characteristics</th>
<th>Market context</th>
<th>Type of hedged risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>PoV</td>
<td>Asia</td>
<td>Large order size (more than 10% of ADV: Average daily consolidated volume)</td>
<td>Possible negative news</td>
<td>Do not miss the rapid propagation of an unexpected news event (especially if I have the information)</td>
</tr>
<tr>
<td>VWAP / TWAP</td>
<td>Asia and Europe</td>
<td>Medium size (from 5 to 15% of ADV)</td>
<td>Any “unusual” volume is negligible</td>
<td>Do not miss the slow propagation of information in the market</td>
</tr>
<tr>
<td>Implementation Shortfall (IS)</td>
<td>Europe and US</td>
<td>Small size (0 to 6% of ADV)</td>
<td>Possible price opportunities</td>
<td>Do not miss an unexpected price move in the stock</td>
</tr>
<tr>
<td>Liquidity Seeker</td>
<td>US (Europe)</td>
<td>Any size</td>
<td>The stock is expected to “oscillate” around its “fair value”</td>
<td>Do not miss a liquidity burst or a relative price move on the stock</td>
</tr>
</tbody>
</table>

More on all this in the three “reference books” for practitioners:
- Market Microstructure in Practice [Lehalle et al., 2013]
- The Financial Mathematics of Market Liquidity [Guéant, 2016]
- Algorithmic and High-Frequency Trading [Cartea et al., 2015]
- Quantitative Trading: Algorithms, Analytics, Data, Models, Optimization [Guo et al., 2016]
The first papers [Almgren and Chriss, 2000], [Bertsimas and Lo, 1998], focussed on the optimal trading rate, or trading speed (i.e. how many shares to buy or sell every 5 minutes) for long metaorders.

- it does not deal with microscopic orderbook dynamics,
- it is a convenient way to take into account any information or constraint at this time scale.

It is very useful for asset managers, brokers, or hedgers. I.e. especially when the decision step is separated from the execution step.
Nevertheless it can be used for opportunistic trading too, when risk management at an intraday scale is important.
The usual (simplistic) example of (continuous time) optimal trading (for a large sell order)

1. Write the Markovian dynamics or the price $P$, the quantity to trade $Q$ and the cash account $X$ for a sell of $Q_0$ shares before $t = T$ (control is the –negative– trading speed $\nu$)

$$dQ = \nu \, dt, \quad dX = -\nu \,(P + \kappa \cdot \nu) \, dt, \quad dP = \mu \, dt + \sigma \, dW.$$ 

2. Write the cost function to maximize

$$V(t, p, q, x, \nu) = \mathbb{E} \left( X_T + Q_T (P_T - A \cdot Q_T) - \phi \int_{\tau = t}^{T} Q_T^2 \, d\tau \right| \mathcal{F}_t).$$

3. it gives the HJB and its terminal condition $V(T, \ldots) = x + q(p - Aq)$

$$-\mu \partial_P V = \partial_t V + \frac{\sigma^2}{2} \partial_P^2 V - \phi \, q^2 + \max_{\nu} \{ \nu \partial_Q V \, dt - \nu (p + \kappa \cdot \nu) \partial_X V \}.$$
4. After the change of variable \( V(t, p, q, x) = x + q p + v(t, q) \), you have

\[-\mu \partial_p V = \partial_t v - \phi q^2 + \max_{\nu} \left\{ \nu \partial_Q v - \kappa \nu^2 \right\}.\]

5. The optimal control is \( \nu^* = \partial_Q v / (2\kappa) \), and the PDE \(-\mu x = \partial_t v - \phi q^2 + \kappa(\partial_Q v)^2 / (4\kappa)\).

6. The value function is quadratic: \( v(t, q) = h_0(t) + q h_1(t) - q^2 h_2(t) / 2 \), you can separate the PDE in three:

\[
\begin{cases}
2\kappa \phi &= -2\kappa h'_2 + h^2_2 \\
-\mu &= h'_1 - 2h_1 h_2 \\
0 &= h'_0 + h^2_1
\end{cases}
\]

And terminal conditions \( h_0(T) = h_1(T) = 0 \) and \( h_2(T) = -2A \): **backward dynamics**.

Cartea and Jaimungal (with misc. co-authors) developed this framework for plenty versions: with a (slightly) different objective function (VWAP, PoV), with permanent market impact \( \mu \to \mu + \nu \), with \( \mu_t \) any (adapted) process, etc.
Two areas are not explored enough

- for practitioners: statistical learning; how to adapt online to regime switches (remember what we said about liquidity game vs. price game)? How to be robust to transitory phases? “Closing the loop” with learning is mixing exploration and exploitation.

- for regulators: game theory; what is the result of putting rational agents together? The more quants will read the four books, the more it will be needed to understand such interactions, and how changing “meta parameters” (ie rules) will modify the outcome of this game?

For game theory on financial market:

- few agents usually leads to principal - agent problems,
- a lot of agents usually leads to mean field games.

Moreover, game theory is a way to obtain robust control.
Mean Field Games

- the number of players needs to be "large enough"
- all players contribute to a "mean field" (i.e. a global variable: available shares, volatility, resource, etc)
- a function of this mean field (at least its mean, may be its standard deviation, etc) appear in this utility function of the players
→ the name on the player cannot be used, but they can have a parameter (like a time horizon or risk aversion) of their own

The methodology is similar to the one to solve static Nash games:
- express the solution (for one agent) and find the solution as if the mean field was known
- you obtain a backward pde
- combine what you know about the mean field to find its forward pde

Liquidity is typically a mean field: the state of the inventory of participants influence their costs and can lead to fire sales [Carmona et al., 2013]. What practitioners call "velocity" of the liquidity (the flows) is a mean field too, it probably forms the prices along with market impact.
A continuum of agents trade optimally “à la Cartea-Jaimungal”.

\[ dS_t = \alpha \mu_t \, dt + \sigma \, dW_t. \]

(1)

\[ dQ_t^a = \nu_t^a \, dt, \]

now for a seller, \( Q_0^a > 0 \) (the associated control \( \nu^a \) will be mostly negative) and the wealth suffers from linear trading costs driven by \( \kappa \) (or temporary, or immediate market impact):

\[ dX_t^a = -\nu_t^a (S_t + \kappa \cdot \nu_t^a) \, dt. \]

Same equations as for the standard framework, except the trend is made of the permanent impact of all agents:

\[ \mu = \int_{a \in \mathcal{A}} \nu^a \, df(a), \]

where \( f(a) \) is the density of the agents in a feature space \( \mathcal{A} \).
The cost function of investor \( a \) selling from \( t = 0 \) and \( T \) is similar to the ones used in [Cartea et al., 2015]: the terminal inventory is penalized and a quadratic running cost is subtracted:

\[
V_t^a := \sup_\nu \mathbb{E}\left( X_T^a + Q_T^a (S_T - A_T^a \cdot Q_T^a) - \phi^a \int_{s=t}^T (Q_s^a)^2 \, ds \bigg| \mathcal{F}_t \right).
\]

Here we took \( T \) common to all investors, i.e. the end of the trading day.

Our framework is then

- Each agent \( a \) has an initial quantity \( Q_0^a \) to buy (\( Q_0^a < 0 \)) or to sell (\( Q_0^a > 0 \)) we can even have purely opportunistic agents (\( Q_0^a = 0 \)).
- They all start at the open of the trading session \( t = 0 \) and end at the close \( t = T \).
- Each of them maximizes the value of his trades for the day: cash + penalized remaining quantity (by \( A^a \)) - cost of risk (with his own risk aversion \( \phi^a \)).
The associated Hamilton-Jacobi-Bellman is

\[ 0 = \partial_t V^a - \phi^a q^2 + \frac{1}{2} \sigma^2 \partial_s^2 V^a + \alpha \mu \partial_s V^a + \sup_{\nu} \left\{ \nu \partial_Q V^a - \nu (s + \kappa \nu) \partial_x V^a \right\}, \]

with the terminal condition \( V^a(T, x, s, q; \mu) = x + q(s - A^a q). \)

**The usual solution:** Following the Cartea and Jaimungal’s approach, we will use the following ersatz: \( V^a = x + qs + \nu^a(t, q; \mu). \) Thus the HJB on \( \nu \) is

\[ -\alpha \mu q = \partial_t \nu^a - \phi^a q^2 + \sup_{\nu} \left\{ \nu \partial_Q \nu^a - \kappa \nu^2 \right\}, \]

with the terminal condition \( \nu^a(T, q; \mu) = -A^a q^2. \)

The associated optimal feedback / control is straightforward to find:

\[ \nu^a(t, q) = \frac{\partial_Q \nu^a(t, q)}{2\kappa}. \]

⇒ We know that if we have the value function of an agent \( \nu \), we can deduce the associated optimal control.
Distribution of agents is mainly defined by the joint distribution $m(t, dq, da)$ of
- the inventory $Q_t^a$, with known initial values.
- the preferences of the agent: the risk aversion $\phi^a$, and the terminal penalization $A^a$.

The net trading flow $\mu$ driving the trend of the public price at time $t$ reads:

$$
\mu_t = \int_{(q, a)} \nu_t^a(q) m(t, dq, da) = \int_{q, a} \frac{\partial Q v^a(t, q)}{2\kappa} m(t, dq, da).
$$

$\Rightarrow \nu^a$ is an implicit function of $\mu$ (look at the HJB), meaning we will have a fixed point problem to solve in $\mu$.

By the dynamics of $Q_t^a$, the transport of the measure $m(t, dq, da)$ has to follow (continuity equation)

$$
\partial_t m + \partial_q \left( m \frac{\partial Q v^a}{2\kappa} \right) = 0 \text{ with initial condition } m_0 = m_0(dq, da).
$$
Obtaining The Backward-Forward Dynamics

Now we can have side to side:

- the HJB (backward) PDE where we plug the value of \( \mu \);
- the (Forward) transport of the mass of agents \( m \), driven by the aggregation of their instantaneous decisions.

\[
\begin{align*}
\begin{cases}
-\alpha q \int_{(q', a')} \frac{\partial Qv^a(t, q')}{2\kappa} m(t, dq', da') & = \partial_t v^a - \phi^a q^2 + \frac{(\partial Qv^a)^2}{4\kappa} \\
\text{aggregate of all agents} & \\
\partial_t m + \partial_q \left( m \frac{\partial Qv^a}{2\kappa} \right) & = 0
\end{cases}
\end{align*}
\]

Under boundary (resp. initial and terminal) conditions:

\[
\begin{cases}
m(0, dq, da) & = m_0(dq, da), \\
v^a(T, q; \mu) & = -A^a q^2, \quad \forall a.
\end{cases}
\]
Explicit Solution For a Special Case

Same preferences for all agents: $\phi^a \equiv \phi$, $A^a \sim A$

We will need a notation for the aggregated (i.e. net) position of all agents $E(t) = \mathbb{E} [Q_t] = \int q \, m(t, dq)$.

Then we can write:

$$E'(t) = \int q \partial_t m(t, dq) = -\int q \partial q \left( m(t, q) \frac{\partial \nu(t, q)}{2\kappa} \right) dq \quad \text{definition}$$

$$= \int q \frac{\partial \nu(t, q)}{2\kappa} m(t, dq) \quad \text{forward dynamics (transport)}$$

Moreover, $\nu(t, q)$ can be expressed as a quadratic function of $q$: \( \nu(t, q) = h_0(t) + q \, h_1(t) - q^2 \frac{h_2(t)}{2} \), leading to:

$$E'(t) = \int q \, m(t, q) \left( \frac{h_1(t)}{2\kappa} - \frac{h_2(t)}{2\kappa} \cdot q \right) dq = \frac{h_1(t)}{2\kappa} - \frac{h_2(t)}{2\kappa} \cdot E(t).$$

In a more compact form:

$$2\kappa E'(t) = h_1(t) - E(t) \cdot h_2(t).$$
1 Standard Algorithmic Trading
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We now collect all the equations:

\begin{align*}
(3a) & \quad 4\kappa \phi = -2\kappa h_2'(t) + (h_2(t))^2, \\
(3b) & \quad \alpha h_2(t) E(t) = 2\kappa h_1'(t) + h_1(t) (\alpha - h_2(t)), \\
(3c) & \quad - (h_1(t))^2 = 4\kappa h_0'(t), \\
(3d) & \quad 2\kappa E'(t) = h_1(t) - h_2(t) E(t).
\end{align*}

with the boundary conditions \( h_0(T) = h_1(T) = 0, \ h_2(T) = 2A, \ E(0) = E_0, \) where \( E_0 = \int q m_0(q) dq \) is the net initial inventory of market participants (i.e. the expectation of the initial density \( m \)).

The Main Equation For Identical Preferences

The previous system of ordinary differential equations implies

\begin{equation}
0 = 2\kappa E''(t) + \alpha E'(t) - 2\phi E(t)
\end{equation}

with boundary conditions \( E(0) = E_0 \) and \( \kappa E'(T) + AE(T) = 0 \).
For any $\alpha \in \mathbb{R}$, the problem (4) has a unique solution $E$, given by

$$E(t) = E_0 a \left( \exp\{r_+ t\} - \exp\{r_- t\} \right) + E_0 \exp\{r_- t\}$$

where $a$ is given by

$$a = \frac{\left( \alpha / 4 + \kappa \theta - A \right) \exp\{-\theta T\}}{-\frac{\alpha}{2} \text{sh}\{\theta T\} + 2 \kappa \theta \text{ch}\{\theta T\} + 2 A \text{sh}\{\theta T\}},$$

the denominator being positive and the constants $r_+^\pm$ and $\theta$ being given by

$$r_\pm := -\frac{\alpha}{4\kappa} \pm \theta, \quad \theta := \frac{1}{\kappa} \sqrt{\kappa \phi + \frac{\alpha^2}{16}}.$$
Solving the Control

Solving \( h_2(t) \)

\( h_2 \) solves the following backward ordinary differential equation (3a): \( 0 = 2\kappa \cdot h'_2(t) + 4\kappa \cdot \phi - (h_2(t))^2 \) under \( h_2(T) = 2A \). It is easy to check the solution is

\[
\begin{aligned}
    h_2(t) &= 2\sqrt{\frac{\kappa \phi}{1 + c_2 \cdot e^{rt}}} \\
    c_2 &= -\frac{1 - A/\sqrt{\kappa \phi}}{1 + A/\sqrt{\kappa \phi}} \cdot e^{-rT}.
\end{aligned}
\]

where \( r = 2\sqrt{\phi/\kappa} \) and \( c_2 \) solves the terminal condition. Hence

\[

\text{Keep in mind the optimal control is}
\[
\nu^* = \frac{\partial_Q \nu(t, q)}{2\kappa} = \frac{h_1(t)}{2\kappa} - \frac{h_2(t)}{2\kappa},
\]

Solving \( h_1(t) \)

The affine component of the control can be easily deduced from \( h_2(t) \) and \( E(t) \):

\[
    h_1(t) = 2\kappa \cdot E'(t) + h_2(t) \cdot E(t).
\]
1. Standard Algorithmic Trading

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For heterogenous preferences (i.e. each agent has his own $\phi^a$ and $A^a$), it is not very different from the upper case, we have similar dynamics for each type of agent $a$:

\[
\begin{align*}
0 &= -2\kappa (h_2^a)' - 4\kappa \phi^a + (h_2^a)^2, \\
2\kappa \alpha \mu(t) &= -2\kappa (h_1^a)' + h_1^a h_2^a, \\
-(h_1^a)^2 &= 4\kappa (h_0^a)',
\end{align*}
\]

We take $A^a = \sqrt{\phi^a \kappa}$ to have a short writing of the solutions and set $\theta^a = h_2^a/(2\kappa)$. We can define a $E^a$ too, and $\mu$ is just the average over all agents:

\[
\mu(t) = \frac{1}{2\kappa} \left( \int_a h_1^a(t) \bar{m}_0(da) - \int_a h_2^a(t) E^a(t) \bar{m}_0(da) \right).
\]

And we show this expression of $\mu$ is regular enough so that:

- for $\alpha$ small enough: there exists a unique solution to the game (thanks to a fixed point theorem),
- there is even enough room for learning.
Setup:
- rounds are days, then number of rounds $n$ increases
- $m(0, da, dq)$ stays (almost) the same every day

Agents:
- each agent has his own view on the aggregated trading speed $\mu \sim \mu^{a,n}$
- he implements the “optimal” strategy using $\mu^{a,n}$ in place of the true $\mu$
- the effective aggregation of speeds is then $m^{n+1}$ during this round
- each agent estimates it with an error
- and updates his view for the next round using a “memory” parameter:

$$
\mu^{a,n+1}(t) := (1 - \pi^{a,n+1}) \mu^{a,n}(t) + \pi^{a,n+1}(m^{n+1}(t) + \epsilon^{a,n+1}(t))
$$

where $\epsilon$ is a bounded estimation error.
When agent $a$ “believes” in $\mu^{a,n}$:

$$h_{1}^{a,n}(t) = \alpha \int_{t}^{T} e^{\theta^{a}(t-s)} \mu^{a,n}(s) \, ds, \quad h_{2}^{a} = 2\kappa \theta^{a}.$$  

And the aggregation of “trading speeds” at step $m + 1$ is

$$m^{n+1}(t) = \frac{1}{2\kappa} \left\{ \int_{a} h_{1}^{a,n}(t) \tilde{m}_{0}(da) - \int_{a} h_{2}^{a,n} E^{a,n}(t) \tilde{m}_{0}(da) \right\}.$$  

That can be written to see the current estimation of agent $a$:

$$m^{n+1}(t) = \frac{\alpha}{2\kappa} \left\{ \int_{t}^{T} ds \int_{a} \mu^{a,n}(s) e^{\theta^{a}(t-s)} \tilde{m}_{0}(da) - \int_{0}^{t} d\tau \int_{\tau}^{T} ds \int_{a} \mu^{a,n}(s) e^{\theta^{a}(2\tau-t-s)} \theta^{a} \tilde{m}_{0}(da) \right\}$$

$$- \int_{a} \theta^{a} e^{-\theta^{a}t} E^{a}_{0} \tilde{m}_{0}(da).$$
Plug the previous equation in

\[ \mu^{a,n+1}(t) := (1 - \pi^{a,n+1}) \mu^{a,n}(t) + \pi^{a,n+1} (m^{n+1}(t) + \epsilon^{a,n+1}(t)), \]

and express

\[ \sup_a ||\mu^{a,n} - \mu||_\infty \leq \cdots. \]

Provided \( \pi^{a,n} \) goes to zero fast enough \((1/n)\), you obtain a law of large numbers:

**Convergence of the incomplete information game**

Within such a setup, this learning game converges towards its perfectly informed version:

\[ \limsup \sup_a ||\mu^{a,n} - \mu||_\infty \leq C ||\epsilon^{a,n}||_\infty. \]

It is probably possible to obtain a CLT...
Outline

1. Standard Algorithmic Trading
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Dependence of the Solution to the Mean Field

The optimal control is

$$\nu^* = \frac{\partial Q \nu(t, q)}{2\kappa} = \frac{h_1(t)}{2\kappa} - q \cdot \frac{h_2(t)}{2\kappa}.$$ 

- The second term is proportional to your inventory, i.e; the remaining quantity to buy/sell, it is independent of $E$;
- The first term embeds the dependence to the mean field: $h_1(t) = 2\kappa \cdot E'(t) + h_2(t) \cdot E(t)$.

$\Rightarrow$ locally you adapt your behaviour to the mean field via $h_1$,
$\rightarrow$ then (you changed your inventory), you slowly (re)adapt to be ready for boundary conditions / costs.
Dynamics of $E$ (left) and $-h_1$ and $h_2$ (right) for a standard set of parameters: $\alpha = 0.4$, $\kappa = 0.2$, $\phi = 0.1$, $A = 2.5$, $T = 5$, $E_0 = 10$. 
Comparison of the dynamics of $E$ (left) and $-h_1$ and $h_2$ (right) between the “reference” parameters of Figure ?? and smaller $\alpha$ (i.e. $\alpha = 0.1$ instead of 0.4) such that $|h_1(0)|$ is smaller.
Comparison of the dynamics of $E$ (left) and $-h_1$ and $h_2$ (right) between the “reference” parameters of Figure ?? and when $\sqrt{\kappa\phi} \sim A$: in such a case $h_2$ is almost constant but $E$ and $h_1$ are almost unchanged.
A specific case for which $E$ is not monotonous: $\alpha = 0.01$, $\kappa = 1.5$, $\phi = 0.03$, $A = 2.5$, $T = 5$ and $E_0 = 10$. 
Numerical explorations of $t^m$ for different values of $\phi$ (very small $\phi$ at the top left to small $\phi$ at the bottom right) on the $\alpha \times \kappa$ plane, when $T = 5$ and $A = 2.5$. The color circles codes the value of $t^m$: small values (dark color) when $E$ changes its slope very early; large values (in light colors) when $E$ changes its slope close to $T$. 

CA Lehalle
Conclusion on MFG of Controls For Liquidation

It is a proof of maturity of the use if stochastic control in financial math:

▶ Four years ago, it was difficult to think about a game theoretical version of the Almgren and Chriss optimal liquidation problem (schied and jaimungal).
▶ Our understanding of the problem itself improved (see Guéant and Cartead and Jaimungal books)
▶ and some extensions of MFG have been needed (see the paper).
▶ but we now know how to handle it (and in a specific case it is fully solved)

Solving game theoretical versions of what we know is important (instead of sophisticating it in a mean field game), because

▶ it is a way to obtain robust control
▶ it helps regulator to understand the system to adjust some meta parameters (κ is this example)

MFG is not the only way to answer to such questions. Nevertheless in general Mean Field Games can take into account interactions between different market participants as soon as they interact via liquidity (i.e. the mean field).
Moreover learning should not be forgot (done in our paper): what does change when information is not complete?


